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Degree condition for the existence of a k-factor containing a given Hamiltonian cycle

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ABSTRACT

Let k be an integer with $k \geq 2$ and let G be a graph having sufficiently large order n. Suppose that kn is even, the minimum degree of G is at least k and $\max\{d_G(x), d_G(y)\} \geq (n+\alpha)/2$ for each pair of nonadjacent vertices x and y in G, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Then G has a k-factor (i.e. a k-regular spanning subgraph) which contains a given Hamiltonian cycle C if $G - E(C)$ is connected.

1. Introduction and results

Graphs considered here are finite undirected graphs without loops and multiple edges. For notation and terminology not defined here we refer the reader to [1].

Let $G = (V, E)$ be a simple graph. We use $V(G)$, $E(G)$, $\delta(G)$ to denote the vertex set, edge set, *minimum degree* in G, respectively. The order of G is $|G| = |V| = n$ and its size is $e(G) = |E|$. A graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle. For $x \in V(G)$, we denote by $N(x)$ the neighborhood of x in G, by $d(x)$ the degree of x in G. For $S \subseteq V(G)$, let $|S|$ denote the number of the vertices in S and $G[S]$ be the subgraph induced by S. In particular, we write $G - S$ for $G[V(G) \setminus S]$. Let H be a subgraph of G and $u \in V(G)$ a vertex, $N(u, H)$ is the set of neighbors of u contained in H. We write $d(u, H) = |N(u, H)|$. Clearly, $d(u, G)$ is the degree of u in G, and we write $d(x)$ to replace $d(x, G)$. If there is no fear of confusion, we often identify a subgraph H of G with its vertex set $V(H)$. For any two vertex-disjoint subgraphs of G, say G_1 and G_2, the *join* $G^\star$ of these two subgraphs is defined as follows: $V(G^\star) = V(G_1) \cup V(G_2)$ and $E(G^\star) = E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1), y \in V(G_2)\}$. We use $G_1 \otimes G_2$ to denote the join of G_1 and G_2. We define the distance $d(x, y)$ between two vertices x and y as the minimum of the lengths of the $x - y$ paths of G. Given a disjoint subset $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of edges in G joining a vertex in A to that in B. For a positive integer k, a *k-factor* is a spanning subgraph F such that $d_F(x) = k$ for each $x \in V(G)$.

A classic sufficient degree condition for a Hamiltonian graph was obtained by Ore.

Theorem 1 (*Ore [2]*). *If $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

Fan (see [3]) extended Theorem 1 by providing an improved sufficient condition for a Hamiltonian graph.

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Theorem 2 (*Fan [3]*). *Let G be a 2-connected graph of order $n \geq 3$. If for any two vertices x and y of G such that $d(x, y) = 2$, $\max\{d(x), d(y)\} \geq n/2$, then G has a Hamiltonian cycle.*

Theorem 3 (*Iida and Nishimura [4]*). *Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 4k - 5$. Suppose that kn is even, $\delta(G) \geq k$, and $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$. Then G has a k-factor.*

Theorem 4 (*Nishimura [5]*). *Let G be a connected graph of order $n \geq 4k - 3$, where $k \geq 3$. Suppose that kn is even, $\delta(G) \geq k$, and $\max\{d(x), d(y)\} \geq n/2$ for each pair of nonadjacent vertices $u, v \in V(G)$. Then G has a k-factor.*

Recently, H. Matsuda [6] proved the following result.

Theorem 5 (*Matsuda [6]*). *Let $k \geq 2$ be an integer and let G be a graph of order $n > 8k^2 - 2(\alpha + 12)k + 3\alpha + 16$, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Suppose that kn is even and the minimum degree $\delta(G)$ of G is at least k. If for any nonadjacent vertices x and y of G, $d(x) + d(y) \geq n + \alpha$, then G has a k-factor containing a given Hamiltonian cycle.*

In this paper, we improve the result of Theorem 5 by the following Theorem:

Theorem 6. *Let $k \geq 2$ be an integer and let G be a graph of order $n > 12(k-2)^2 + 2(5-\alpha)(k-2) - \alpha$. Suppose that kn is even, $\delta(G) \geq k$ and $\max\{d(x), d(y)\} \geq (n + \alpha)/2$ for each pair of nonadjacent vertices x and y in G, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Then G has a k-factor which contains a given Hamiltonian cycle C if $G - E(C)$ is connected.*

2. Main result

We list Tutte's Theorem regarding the existence of a k-factor which will be used in our proof of the main result.

Theorem 7 (*Tutte [7]*). *Let G be a graph and $k \geq 1$ an integer. Then G has a k-factor if and only if*

$$\theta_G(S, T, k) = k|S| + \sum_{x \in T} (d_{G-S}(x) - k) - h_G(S, T, k) \geq 0$$

for all disjoint subsets S and T of $V(G)$, where $h_G(S, T, k)$ is the number of the components D of $G - (S \cup T)$ such that $k|D| + e_G(V(D), T) \equiv 1 \pmod 2$. Furthermore, whether G has a k-factor or not, we have that $\theta_G(S, T, k) \equiv k|V(G)| \pmod 2$ for any disjoint subsets S and T of $V(G)$.

We call such a component D *odd component*. Let $\omega(G)$ denote the number of components of G and let $o(G)$ denote the number of components of G each of which has odd order. We provide several propositions.

Proposition 1. *If the condition of Theorem 6 is satisfied, $k \geq 3$ and $G - E(C)$ is connected for any Hamiltonian cycle C in G. Then $\omega(G - E(C) - A) \leq |A| + 1$ for every $A \subset V(G)$ or $k = 3$ and G has a 3-factor which contains C.*

Proof. Let $H := G - E(C)$. Suppose there exists a set $A \subset V(G)$ such that $\omega(H - A) \geq |A| + 2$. If $A = \emptyset$, it is obvious. Hence, we may assume that $A \neq \emptyset$. Let $C_1, C_2, \ldots, C_\omega$ be the components of $H - A$. We assume that $|C_1| \leq |C_2| \leq \cdots \leq |C_\omega|$. Since H is connected and $\omega \geq 2$ by the assumption, there exists a vertex $x_i \in V(C_i)$ with $N(x_i) \cap A \neq \emptyset$ for every $i \in \{1, 2, \ldots, \omega\}$.

We claim that there exist $x_j, x_l \in \{x_1, x_2, \ldots, x_{|A|+1}\}$ such that x_j and x_l are nonadjacent. Otherwise, we may assume that all the vertices of $V(C_1)$ are adjacent to those of $V(C_2)$ in C and $|C_1| \leq |C_2| \leq 2$. Suppose that $|A| \geq 2$, then $\omega(H - A) \geq 4$. If $|C_2| = 2$, then for each $u_1 \in V(C_1)$, $e_C(u_1, V(C_2)) = 2$ and so $u_1u_3 \notin E(G)$ for each $u_3 \in V(C_3)$. If $|C_2| = 1$, then $u_1u_2 \in E(G)$ for $\{u_1\} = V(C_1)$ and $\{u_2\} = V(C_2)$. Consequently, there exists $u_3 \in V(C_3)$ such that $u_1u_3 \notin E(C)$ or $u_2u_3 \notin E(C)$ and we are done. Hence, it remains the case that $|A| = 1$ and $\omega(H - A) = 3$. Clearly, $|C_1| = 1$ and $|C_2| \leq 2$, denote $A = \{z\}$. If $|C_2| = 1$, then $V(C_2) = \{x_2\}$. By the symmetry role of x_1 and x_2, there exist two types of configurations of G along C, see Figs. 1 and 2, where $V(C_3) = V(P) \cup \{u\}$. If $|C_2| = 2$, denote $V(C_2) = \{x_2, x_2'\}$. Consequently, there exist three types of configurations according to whether $zx_2' \in E(G)$ or $zx_2' \notin E(G)$, where $V(C_3) = V(P) \cup \{u\}$, see Fig. 3. Note that in either case, $k = 3$ since $d_G(x_1) = 3$, then $\alpha = 3$. Furthermore, Type 2 will not appear. As n is even, then $\max\{d_H(w_1), d_H(w_2)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices $w_1, w_2 \in V(H)$, applying Theorem 2, $G - E(C)$ contains a Hamiltonian cycle and so G contains two edge-disjoint Hamiltonian cycles. Consequently, $G - E(C)$ contains a 1-factor since n is even, denoted by $\mathcal{F}$. Then, $E(C) \cup \mathcal{F}$ forms a desired 3-factor which contains C.

From the above claim, by the condition of Theorem 6, at least one of the two vertices, say x_j, has degree at least $\frac{n+\alpha-4}{2}$ in H. This yields

$$\frac{n + \alpha - 4}{2} \leq d_H(x_j) \leq |V(C_j)| - 1 + |A|.$$

Therefore, for every $i \geq j$,

$$|V(C_i)| \geq \frac{n + \alpha - 4}{2} - |A| + 1.$$

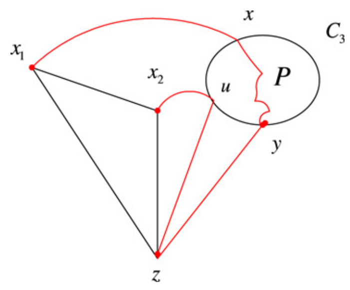

Fig. 1. Type 1.

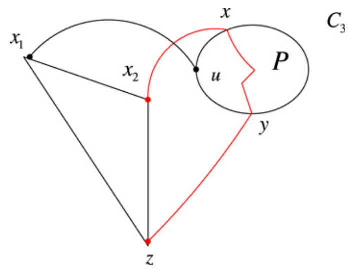

Fig. 2. Type 2.

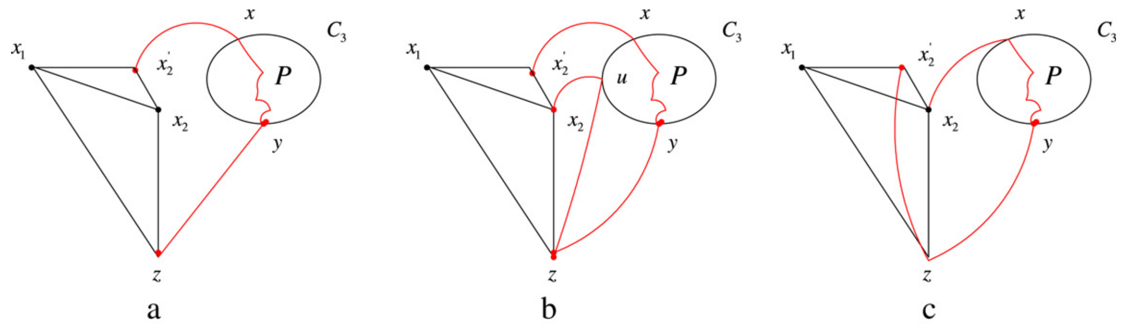

Fig. 3. Three configurations along C.

Since $j \leq |A| + 1$ and $3 \leq \alpha \leq 4$, we obtain

$$n \geq |A| + \sum_{i=1}^{|A|} V(C_i) + \sum_{i=|A|+1}^{|A|+2} V(C_i) \geq 2|A| + 2\left(\frac{n+\alpha-4}{2} - |A| + 1\right) = n + 1,$$

a contradiction. □

Proposition 2. *Let G be a graph satisfying the condition of Theorem 6. Suppose that for each Hamiltonian cycle C in G, $H = G - E(C)$ is connected and $k \geq 3$. Then G has a 3-factor which contains any given Hamiltonian cycle C.*

Proof. Suppose for a contradiction, for a given Hamiltonian C in G, G does not contain a 3-factor which contains C, this implies $G - E(C)$ does not contain a 1-factor. By Tutte's 1-factor Theorem [8], there exists a set $A \subset V(G)$ such that $o(H - A) > |A|$. Since $3n$ is even by the condition of Theorem 6, then n is even. It follows that $o(H - A) \geq |A| + 2$, which contradicts Proposition 1. □

Proposition 3. *Let G be a graph satisfying the condition of Theorem 6. Suppose that for each Hamiltonian cycle C in G, $H = G - E(C)$ is connected and $k \geq 3$. Then $\omega(H - A) \leq |A| + 5 - \alpha$ for all $\emptyset \neq A \subseteq V(H)$ or $k = 3$ and G has a 3-factor which contains C.*

Proof. It is obvious by applying Proposition 1. □

Proof of Theorem 6. We prove it by contradiction. We may assume that $k \geq 3$ since G contains a Hamiltonian cycle by Theorem 2. Let $H := G - E(C)$, $\rho := k - 2$ where $k \geq 3$. In the following, we may assume that for each Hamiltonian C,

$G - E(C)$ is connected. Clearly, $V(H) = V(G)$ and $\rho \geq 1$,

$$d_H(x) = d(x) - 2 \geq \rho \quad \text{for all } x \in V(H), \tag{1}$$

and since C is a Hamiltonian cycle of G, we have

$$\max\{d_H(u), d_H(v)\} \geq \frac{n+\alpha-4}{2} \quad \text{for each nonadjacent } u, v \in V(H). \tag{2}$$

Obviously, G has the desired factor if and only if H has a ρ-factor. Suppose to the contrary, H does not contain such a factor, then by Theorem 7, there exist disjoint subsets S and T of $V(H)$ such that

$$\theta_H(S, T, \rho) = \rho|S| + \sum_{x \in T}(d_{H-S}(x) - \rho) - h_H(S, T, \rho) \leq -2. \tag{3}$$

We choose such subsets S and T such that $|T|$ is minimum whereas $|S|$ is maximal with respect to $|T|$. Further, we choose ρ as small as possible.

In view of Proposition 2, we see that H contains 1-factor. So in the following proof we assume that $\rho \geq 2$.

Claim 1. $|T| \geq |S| + 1$.

Proof. We first show that H has a $(\rho - 2)$-factor. It is trivial if $\rho = 2$. So we assume that $\rho \geq 3$. By the choice of ρ, H has a $(\rho - 2)$-factor, by Theorem 7, we have $\theta_H(S, T, \rho - 2) \geq 0$. Note that $h_H(S, T, \rho) = h_H(S, T, \rho - 2)$ by the definition of odd components, we obtain $-2 \geq \theta_H(S, T, \rho) - \theta_H(S, T, \rho - 2) = 2|S| - 2|T|$, which implies that $|T| \geq |S| + 1$. □

By Claim 1, it is clear that $T \neq \emptyset$.

Claim 2. $2|S| \leq n - 6\rho + \alpha - 3$.

Proof. Suppose that $2|S| \geq n - 6\rho + \alpha - 2$, that is, $n - 2|S| \leq 6\rho - \alpha + 2$. Thus $|T| - |S| = n - 2|S| - |V(H) - (S \cup T)| \leq 6\rho - \alpha + 2 - h_H(S, T, \rho)$. By (3), we obtain that

$$\begin{aligned}
\sum_{x \in T} d_{H-S}(x) &\leq \rho|T| - \rho|S| + h_H(S, T, \rho) - 2 \\
&\leq \rho(|T| - |S|) + h_H(S, T, \rho) - 2 \\
&\leq \rho(6\rho - \alpha + 2 - h_H(S, T, \rho)) + h_H(S, T, \rho) - 2 \\
&\leq \rho(6\rho - \alpha + 2) - 2.
\end{aligned}$$

Since $n > 12\rho^2 + 2(5 - \alpha)\rho - \alpha$, we have $|T| \geq |S| + 1 \geq \frac{n}{2} - 2\rho + \frac{\alpha}{2} > 6\rho^2 + (3 - \alpha)\rho > 2$. Consequently, together with Claim 1, we have

$$\begin{aligned}
\frac{\sum_{x \in T} d_{H-S}(x)}{|T| - 2} &\leq \frac{\rho(6\rho - \alpha + 2) - 2}{|T| - 2} \\
&\leq \frac{\rho(6\rho - \alpha + 2) - 2}{|S| - 1} \\
&\leq \frac{2\rho(6\rho - \alpha + 2) - 4}{n - 6\rho + \alpha - 4} < 1,
\end{aligned}$$

where the last estimation follows from $n > 12\rho^2 + 2(5 - \alpha)\rho - \alpha$. It follows that

$$\sum_{x \in T} d_{H-S}(x) \leq |T| - 3. \tag{4}$$

Let $T_0 = \{x \in T | d_{H-S}(x) = 0\}$. It follows from Claim 1 that $n \geq |S| + |T| \geq 2|S| + 1$. Hence if n is even, $n \geq 2|S| + 2$. By the definition of α and the assumption that kn is even, $n \geq 2|S| + 2$ if $\alpha = 3$ and $n \geq 2|S| + 1$ if $\alpha = 4$. In either case, we have $d_H(x) \leq |S| < \frac{n+\alpha-4}{2}$ for any $x \in T_0$. Note that $|T_0| \geq 3$ by inequality (4), there exist two vertices $x, y \in T_0$ such that $xy \notin E(G)$. Hence, we obtain that

$$\frac{n+\alpha}{2} \leq \max\{d(x), d(y)\} \leq |S| + |N_C(x)| < \frac{n+\alpha-4}{2} + 2 < \frac{n+\alpha}{2},$$

which contradicts Claim 1. □

Claim 3 (*H. Matsuda, [6]*). *For any $x \in T$, $d_{H-S}(x) \leq \rho - 2 + h_H(S, T, \rho)$ and $d_T(x) \leq \rho - 2$.*

Proof. Let $T' = T \setminus \{x\}$ for any $x \in T$. By the choice of T we have $\theta_H(S, T', \rho) \geq 0$. Since $\theta_H(S, T, \rho) \leq -2$, we obtain $2 \leq \theta_H(S, T', \rho) - \theta_H(S, T, \rho) \leq \rho - d_{H-S}(x) + h_H(S, T, \rho) - h_H(S, T', \rho)$ which implies $d_{H-S}(x) \leq \rho - 2 + h_H(S, T, \rho) - h_H(S, T', \rho)$. This inequality, together with $e_H(x, H - (S \cup T)) \geq h_H(S, T, \rho) - h_H(S, T', \rho)$ yields $d_T(x) = d_{H-S}(x) - e_H(x, H - (S \cup T)) \leq \rho - 2$. □

Claim 4 (*H. Matsuda, [6]*). *$e_H(u, T) \leq \rho - 2$ for any $u \in V(H) \setminus (S \cup T)$.*

Proof. By the maximality of S, $\theta_H(S \cup \{u\}, T, \rho) \geq 0$ for any $u \in V(H) \setminus (S \cup T)$. Note that $\theta_H(S, T, \rho) \leq -2$, we have $2 \leq \theta_H(S \cup \{u\}, T, \rho) - \theta_H(S, T, \rho) \leq \rho - e_H(u, T)$, which implies that $e_H(u, T) \leq \rho - 2$. □

For the sake of convenience, let $C_1, C_2, \ldots, C_\omega$ be the odd components of $H - (S \cup T)$ with $|C_1| \leq \cdots |C_\omega|$, where $\omega = h_H(S, T, \rho)$ is the number of odd components of $H - (S \cup T)$.

Claim 5. *For $x \in V(C_i)$, if $d(x) \geq \frac{n+\alpha}{2}$, then $2|C_i| \geq n - 2|S| - 2\rho + \alpha + 2$.*

Proof. By Claim 4, we have

$$\begin{aligned} \frac{n+\alpha}{2} \leq d(x) &\leq e_H(x, T) + |S| + |C_i| - 1 + |N_C(x)| \\ &\leq \rho + |S| + |C_i| - 1. \end{aligned}$$

Hence $2|C_i| \geq n - 2|S| - 2\rho + \alpha + 2$. □

Claim 6. *$\omega = h_H(S, T, \rho) \leq 4$.*

Proof. Suppose that $\omega = h_H(S, T, \rho) \geq 5$. There are no two nonadjacent vertices $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$. Otherwise by the assumption of G, we assume that $d(u_i) \geq \frac{n+\alpha}{2}$ where $i = 1$ or 2. So $2|C_i| \geq n - 2|S| - 2\rho + \alpha + 2$ ($i = 1$ or 2) by Claim 5. By Claim 2, we obtain

$$\begin{aligned} n &\geq |S| + |T| + |C_1| + |C_2| + |C_3| + |C_4| + |C_5| \\ &\geq 2|S| + 1 + (6 - i)|C_i| \\ &\geq 2|S| + 1 + 2(n - 2|S| - 2\rho + \alpha + 2) \\ &> n + 1, \end{aligned}$$

a contradiction. Thus each vertex of $V(C_1)$ is adjacent to each vertex of $V(C_2)$ in G and so is in C. Then we have $|C_1| \leq |C_2| \leq 2$. If $|C_2| = 2$, then for each $u_1 \in V(C_1)$, $e_C(u_1, V(C_2)) = 2$ and thus $u_1u_3 \notin E(C)$ for each vertex $u_3 \in V(C_3)$. If $|C_2| = 1$, then $u_1u_2 \in E(C)$ for $\{u_1\} = V(C_1)$ and $\{u_2\} = V(C_2)$. Thus there exists a vertex $u_3 \in V(C_3)$ such that at least one of $\{u_1u_3, u_2u_3\}$ is not in $E(C)$. We assume that $u_1u_3 \notin E(C)$. If $d(u_1) \geq \frac{n+\alpha}{2}$, then it is the previous case. If $d(u_3) \geq \frac{n+\alpha}{2}$, by Claim 5, $2|C_3| \geq n - 2|S| - 2\rho + \alpha + 2$. This inequality together with Claim 2 gives

$$\begin{aligned} n &\geq |S| + |T| + |C_1| + |C_2| + |C_3| + |C_4| + |C_5| \\ &\geq |S| + |T| + 2 + 3|C_3| \\ &\geq 2|S| + 1 + 2 + \frac{3}{2} \times (n - 2|S| - 2\rho + \alpha + 2) \\ &> n + 1, \end{aligned}$$

a contradiction again. □

Claim 7. *$G[T]$ is a complete subgraph of G.*

Proof. For $xy \notin E(G)$ where $x, y \in V(T)$, we may assume $d(x) \geq \frac{n+\alpha}{2}$, then by Claims 3 and 6,

$$\frac{n+\alpha}{2} \leq d(x) \leq e(x, S) + d_{H-S}(x) + 2 \leq |S| + \rho - 2 + h_H(S, T, \rho) + 2 \leq \rho + 4 + |S|.$$

Since $\rho \geq 2$, according to Claim 2, $n + \alpha \leq 2(\rho + 4 + |S|) \leq 2\rho + 8 + n - 6\rho + \alpha - 3 = n + \alpha - 4\rho + 5 < n + \alpha$, a contradiction. Thus $xy \in E(G)$. □

Define $m_1 = \min\{d_{H-S}(x) | x \in T\}$ and let $x_1 \in T$ be a vertex with $d_{H-S}(x_1) = m_1$. If $m_1 \geq \rho + 2$, since $|T| \geq 1$, then by Proposition 3, $\theta_H(S, T, \rho) \geq \rho|S| + \sum_{x \in T}(m_1 - \rho) - h_H(S, T, \rho) \geq |S| + |T| - \omega(H - (S \cup T)) + |T| \geq |T| - 2 \geq -1$, which contradicts inequality (3), hence $m_1 \leq \rho + 1$.

By Claims 1, 3 and 7 we have $|S| + 1 \leq |T| \leq d_T(x_1) + |\{x_1\}| + |N_C(x_1)| \leq \rho + 1$ which implies that $|S| \leq \rho$ and $|T| \leq \rho + 1$.

Claim 8. $|C_1| \geq 2$.

Proof. Suppose that $|C_1| = |\{u\}| = 1$. By Claim 4, $e_H(u, T) \leq \rho - 2$. If $e_H(u, T) = \rho - 2$, then $\rho + e_H(u, T) = 2\rho - 2$. This contradicts the fact that C_1 is an odd component of $H - (S \cup T)$, thus we assume $e_H(u, T) \leq \rho - 3$. By (1), $|S| + e_H(u, T) \geq d_H(u) \geq \delta(H) \geq \rho$, then $|S| \geq 3$. If $|T| \leq \rho - 1$, by Claim 6, inequality (3) and the fact $|S| + d_{H-S}(x) \geq \delta(H) \geq \rho$ for every $x \in T$, we have $-2 \geq \theta_H(S, T, \rho) \geq |S| + \sum_{x\in T}(|S| + d_{H-S}(x) - \rho) - h_H(S, T, \rho) \geq |S| - h_H(S, T, \rho) \geq -1$. Hence, a contradiction. So we assume that $|T| \geq \rho$. So $|T| = \rho$ or $\rho + 1$. By Claim 7, we obtain $d_{H-S}(x) \geq d_T(x) \geq \rho - 3$ for each $x \in T$, implying $\rho \geq 3$. If $e_H(u, T) \leq \rho - 4$, then $|S| \geq 4$. Since $G[T]$ is complete by Claim 7, $|E(G[T])| = |T|(|T| - 1)/2$. As C is a Hamiltonian cycle, $|E(G[T]) \cap C| \leq |T| - 1$. Consequently, we obtain that

$$\begin{aligned}\sum_{x\in T} d_{H-S}(x) &\geq 2|E(G[T]) \setminus E(C)| \\ &\geq |T|(|T| - 1) - 2(|T| - 1) = (|T| - 1)(|T| - 2).\end{aligned}$$

Then it follows from Claim 6 and the fact $|S| \geq 4$ that

$$\begin{aligned}\theta_H(S, T, \rho) &= \rho|S| + \sum_{x\in T}(d_{H-S}(x) - \rho) - h_H(S, T, \rho) \\ &\geq \rho|S| + (|T| - 1)(|T| - 2) - \rho|T| - h_H(S, T, \rho) \\ &\geq 4\rho + |T|^2 - (\rho + 3)|T| - 2 \\ &= (|T| - 3)(|T| - \rho) + \rho - 2. \end{aligned} \tag{5}$$

Note that $\rho \geq 3$, if $|T| = \rho$, then $\theta_H(S, T, \rho) \geq 1$, which contradicts (3). Hence, $|T| = \rho + 1$, then $|T| \geq 3$ and by (5), we see that $\theta_H(S, T, \rho) \geq 1$, again a contradiction. Thus, we have $e_H(u, T) = \rho - 3$. If $|T| = \rho + 1$, then by Claim 7, $d_{H-S}(x) \geq \rho - 2$ for each $x \in T$. Since $|E(G[T]) \cap C| \leq |T| - 1$, then $\theta_H(S, T, \rho) \geq 3\rho - (\rho - 1) - 8 = 2\rho - 7 \geq -1$, which contradicts (3). Hence, it remains the case that $|T| = \rho$, by Claim 7, $d_{H-S}(x) \geq \rho - 3$ for each $x \in T$. Recalling that H is connected and $\rho \geq 3$, therefore, we have $\theta_H(S, T, \rho) \geq 3\rho + (\rho - 3)(\rho - 2 - \rho) - 10 = \rho - 4 \geq -1$, which contradicts (3). Hence we complete the proof of Claim 8. □

By Claim 6, $h_H(S, T, \rho) \leq 4$. We will finish the proof by using the following four Claims.

Claim A. $h_H(S, T, \rho) \neq 4$.

Proof. Otherwise, by Claim 8, $2 \leq |C_1| \leq |C_2| \leq |C_3| \leq |C_4|$, thus there exist two vertices $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$ such that $u_1u_2 \notin E(G)$. If $d(u_1) \geq \frac{n+\alpha}{2}$, then by Claims 1, 2 and 5, we have

$$\begin{aligned}n &\geq |S| + |T| + |C_1| + |C_2| + |C_3| + |C_4| \geq 2|S| + 1 + 4|C_1| \\ &\geq 2|S| + 1 + 2(n - 2|S| - 2\rho + \alpha + 2) \\ &> n + 1.\end{aligned}$$

This is a contradiction. Therefore, $d(u_1) < \frac{n+\alpha}{2}$, which implies $d(u_2) \geq \frac{n+\alpha}{2}$, then $2|C_2| \geq n - 2|S| - 2\rho + \alpha + 2$ by Claim 5. This inequality, together with Claims 1, 2 and 8, gives that

$$\begin{aligned}n &\geq |S| + |T| + |C_1| + |C_2| + |C_3| + |C_4| \\ &\geq 2|S| + 1 + 2 + \frac{3}{2} \times (n - 2|S| - 2\rho + \alpha + 2) \\ &> n + 1,\end{aligned}$$

a contradiction. □

Claim B. $h_H(S, T, \rho) \neq 2$.

Proof. We prove it by contradiction. Assume that $h_H(S, T, \rho) = 2$. First we show that $m_1 = \min\{d_{H-S}(x)|x \in T\} \leq \rho$. Otherwise, $m_1 = \rho + 1$, and by (3), $2 = h_H(S, T, \rho) \geq \rho|S| + (m_1 - \rho)|T| + 2$ which implies that $S \cup T = \emptyset$, a contradiction. Therefore, $m_1 \leq \rho$. Since $|S| + d_{H-S}(x_1) \geq \delta(H) \geq \rho$, then $|S| \geq \rho - m_1$. This inequality, together with (3), gives us that

$$\begin{aligned}2 = h_H(S, T, \rho) &\geq \rho|S| + (m_1 - \rho)|T| + 2 \\ &\geq \rho(\rho - m_1) + (m_1 - \rho)|T| + 2 \\ &= (\rho - m_1)(\rho - |T|) + 2 \geq 2.\end{aligned} \tag{6}$$

Since $\rho \geq m_1$, the last inequality holds when $|T| \leq \rho$ unless $|T| = \rho+1$. When $|T| = \rho+1$, by Claim 7, $m_1 \geq |T|-3 = \rho-2$. Clearly, $m_1 \leq \rho - 1$. If $|S| = 0$, then $m_1 = \rho$, then inequality (6) still holds. So, $|S| \geq 1$. If $|S| = 1$, then $m_1 \geq \rho - 1$. By Claim 8 and the minimality of T, we have $\theta_H(S, T, \rho) \geq \rho + (\rho + 1 - 2)(\rho - 1 - \rho) + 2(\rho - \rho) - 2 = \rho - \rho - 1 = -1$, which contradicts (3). Therefore, $|S| \geq 2$. By Claim 8 and the minimality of T, it follows that $\theta_H(S, T, \rho) \geq 2\rho + (\rho + 1 - 3)(\rho - 2 - \rho) + 3(\rho - 1 - \rho) - 2 = -1$, which contradicts (3) again.

Consequently, it follows from (6) that $(\rho - m_1)(\rho - |T|) = 0$. This implies that $|S| + m_1 = \rho$ and $\rho = m_1$ or $\rho = |T|$. Note that $|T| \le m_1 + 1 \le \rho + 1$, $|T| = \rho$ implies that $m_1 = \rho - 1$ or ρ, then $|S| = 1$ or $|S| = 0$. Next we consider the following two cases.

Case (B-1). $|S| = 1$, $|T| = \rho$ and $d_{H-S}(x) = \rho - 1$ for any $x \in T$.

In this case, $\frac{n+\alpha}{2} > d(x) \ge \rho - 1 + |S| + |N_C(x)|$ for any $x \in T$. Since $G[T]$ is a complete subgraph by Claim 7, we have $e_G(T, H - (S \cup T)) = 0$, therefore, for each $u_1 \in V(C_1)$, we obtain

$$\begin{aligned} d(u_1) &\le \frac{n - |S| - |T|}{2} - 1 + |S| + |N_C(u_1)| \\ &= \frac{n + 3 - \rho}{2} < \frac{n+\alpha}{2}. \end{aligned}$$

Consequently, we find a pair of nonadjacent vertices $u_1 \in V(C_1)$ and $x \in G[T]$ such that $\max\{d(u_1), d(x)\} < \frac{n+\alpha}{2}$, a contradiction.

Case (B-2). $|S| = 0$, $m_1 = \rho$, $|T| = \rho$ and $d_{H-S}(x) = \rho$ for any $x \in T$.

Note $\frac{n+\alpha}{2} > d(x) \ge \rho - 1 + |S| + |N_C(x)|$ for any $x \in T$. We have $\theta_H(S, T, \rho) = \rho|S| + \sum_{x\in T}(d_{H-S}(x) - \rho) - 2 = -2$.

Since $m_1 = \rho$, then $\frac{n+\alpha}{2} > d(x_1) = m_1 + |N_C(x_1)| = \rho + 2$. On the other hand, by Claim 7, (3) and the fact $d_{H-S}(x) = \rho$ for each $x \in T$, there is at most one vertex u_i of C_i ($i = 1$ or 2) that satisfies $u_i \in N_H(x)$ ($x \in T$). Since $|C_1| \ge 2$ by Claim 8, we can find another vertex $y \in V(C_i)$ such that $xy \notin E(H)$. Similarly as Case (B-1), we obtain

$$\begin{aligned} d_H(y) &\le \frac{n - |S| - |T|}{2} - 1 + |S| \\ &= \frac{n - 2 - \rho}{2} < \frac{n + \alpha - 4}{2} \end{aligned}$$

which contradicts the inequality (2). $\square$

Claim C. $h_H(S, T, \rho) \ne 3$.

Proof. Otherwise, by Claim 8, $2 \le |C_1| \le |C_2| \le |C_3|$, thus there exist two vertices $u_1 \in V(C_1)$ and $u_2' \in V(C_2)$ such that $u_1u_2' \notin E(G)$. If $d(u_1) \ge \frac{n+\alpha}{2}$, similarly, by Claims 1 and 2, we have

$$\begin{aligned} n &\ge |S| + |T| + |C_1| + |C_2| + |C_3| \ge |S| + |T| + 3|C_1| \\ &\ge 2|S| + 1 + \frac{3}{2} \times (n - 2|S| - 2\rho + \alpha + 2) \\ &> n + 1. \end{aligned} \tag{7}$$

This is a contradiction. So $d(u_1) < \frac{n+\alpha}{2}$, then $d(u_2') \ge \frac{n+\alpha}{2}$. We have $2|C_2| \ge n - 2|S| - 2\rho + \alpha + 2$ by Claim 5. In fact, for any $x \in V(C_1)$, we have $d(x) < \frac{n+\alpha}{2}$ by using the same argument. If there exists $u_1 \in V(C_1)$ such that $u_1x_1 \notin E(G)$, where $d_{H-S}(x_1) = m_1$ and $x_1 \in T$. By the degree condition, $\frac{n+\alpha}{2} \le d(x_1) \le |S| + m_1 + 2 \le 2\rho + 3$ (since $m_1 \le \rho + 1$), then $n \le 2(2\rho + 3) - \alpha \le 4\rho^2 + 3\rho - \alpha$, this inequality contradicts the fact that $n > 12\rho^2 + 2(5 - \alpha)\rho - \alpha$, thus $T \cup C_1 \subseteq N_G(x_1) \cup \{x_1\}$, which gives us that

$$|T \cup C_1| \le |N_G(x_1) \cup \{x_1\}| \le m_1 + 1 + 2 \le \rho + 2 + 2 = \rho + 4. \tag{8}$$

Since $d_{G-S}(x) \le d_{H-S}(x) + 2 \le \rho + 3$ for all $x \in T$. Note $|S| \le \rho$, $|T \cup C_1| \le \rho + 4$, $\rho \ge 2$, $|T| \le \rho + 1$ and $n > 12\rho^2 + 2(5 - \alpha)\rho - \alpha$, we have $n - |S \cup T \cup C_1| > (\rho + 3)(\rho + 1) + 1 \ge \sum_{x\in T} d_{G-S}(x) + 1$. The previous inequality implies that there exists a vertex $u_2 \in V(G) \setminus \{S \cup T \cup C_1\}$ such that $e_G(u_2, T) = 0$. Note that $\rho \ge 2$, by Claim 7 and (8), $d(x) = |T| - 1 \le \rho + 3 < \frac{n+\alpha}{2}$ for each $x \in T$. Thus we may assume that $\frac{n+\alpha}{2} \le d(u_2) \le (|C_m| - 1 + |S| + 2)$ which leads us to that $2|C_m| \ge n + \alpha - 2 - 2|S|$, where C_m is the component such that $u_2 \in V(C_m)$. If $m \ne 2, 3$, then Claims 1 and 2 together with Claim 8 lead to

$$\begin{aligned} n &\ge |S| + |T| + |C_1| + |C_2| + |C_3| + |C_m| \\ &\ge 2|S| + 1 + 2 + (n - 2|S| - 2\rho + \alpha + 2) + \frac{n + \alpha - 2|S| - 2}{2} \\ &= 2|S| + 1 + 2 + n - 2|S| + \rho + \alpha + 2 + \frac{1}{2} \\ &> n + 1. \end{aligned}$$

This contradiction implies that $m = 2$ or 3. In the following, we prove that $|S| = 0$. Otherwise, $|S| \ge 1$. If $m_1 = \rho$ or $\rho + 1$, then it follows that $\theta_H(S, T, \rho) \ge \rho|S| + (m_1 - \rho)|T| - h_H(S, T, \rho) \ge \rho - 3 \ge -1$ since $\rho \ge 2$. This contradicts (3). Thus

$m_1 \le \rho - 1$, by $T \cup C_1 \subseteq N_G(x_1) \cup \{x_1\}$, we have

$$\begin{aligned} |T| + 2 \le |T \cup C_1| &\le d_{H-S}(x_1) + |\{x_1\}| + |N_C(x_1) \cap (T \cup C_1)| \\ &\le m_1 + 1 + 2 \le \rho + 2 \end{aligned} \tag{9}$$

which implies that $|T| \le \rho$. We first assume that $|T| = \rho$. Then $|C_1| = 2$, $m_1 = \rho - 1$ and $|N_C(x_1) \cap (T \cup C_1)| = 2$. If $|S| \ge 2$, then $\theta_H(S, T, \rho) \ge \rho - 3 \ge -1$, a contradiction. So $|S| = 1$, then using a similar argument as in Case (B-1), we arrive at a contradiction to (2). Hence it remains the case that $|T| \le \rho - 1$. If $|T| \le \rho - 2$, then $-2 \ge \theta_H(S, T, \rho) = \rho|S| + \sum_{x\in T}(d_{H-S}(x) - \rho) - h_H(S, T, \rho) \ge 2|S| + \sum_{x\in T}(|S| + d_{H-S}(x) - \rho) - h_H(S, T, \rho) \ge 2|S| - h_H(S, T, \rho) \ge 2|S| - 3 \ge -1$ since $|S| + d_{H-S}(x) - \rho \ge \delta(H) - \rho \ge 0$, which contradicts (3). Hence $|T| = \rho - 1$. Then $-2 \ge \theta_H(S, T, \rho) = |S| + \sum_{x\in T}(|S| + d_{H-S}(x) - \rho) - h_H(S, T, \rho) \ge \sum_{x\in T}(|S| + d_{H-S}(x) - \rho) + |S| - 3 \ge |S| - 3 \ge -2$ which implies that for any $x \in T$, $|S| + d_{H-S}(x) = \rho$ and $|S| = 1$. Particularly, $m_1 = d_{H-S}(x_1) = \rho - |S| = \rho - 1$. By a similar argument as in Case (B-1), we obtain a contradiction.

Without loss of generality, suppose that $u_2 \in V(C_3)$. Since $S = \emptyset$, it follows that $e_G(x, T) > 0$ for any $x \in V(C_3)$. On the other hand, by Claim 3, for any $x \in T$, we have $d_H(x) \le \rho + 1$. Note that $|C_3| \ge \frac{n-2\rho+\alpha+2}{2} > \rho + 1$, $|C_1| \ge 2$ and $T \cup C_1 \subseteq N_G(x_1) \cup \{x_1\}$, then there exists a vertex $v \in C_3$ such that $e_G(v, T) = 0$. This contradicts $e_G(x, T) > 0$ for any $x \in V(C_3)$. Hence we complete the proof of Claim C. □

Claim D. $h_H(S, T, \rho) \not\le 1$.

Proof. Otherwise, by (3), we have

$$-1 \ge h_H(S, T, \rho) - 2 \ge \rho|S| + \sum_{x\in T}(d_{H-S}(x) - \rho). \tag{10}$$

If $|T| \le \rho$, then it follows from (10) that $-1 \ge \rho|S| + \sum_{x\in T}(d_{H-S}(x) - \rho) \ge \sum_{x\in T}(|S| + d_{H-S}(x) - \rho) \ge 0$, a contradiction. Thus we may assume that $|T| = \rho + 1$. If $m_1 \ge \rho$, then by (10), $-1 \ge \rho|S| + \sum_{x\in T}(d_{H-S}(x) - \rho) \ge \rho|S|$, again a contradiction. So $m_1 \le \rho - 1$. Note that $|S| + d_{H-S}(x_1) \ge \delta(H) \ge \rho$, then $|S| \ge \rho - m_1 \ge 1$. Suppose that $|S| = 1$, then $m_1 = \rho - 1$, by the definition of m_1, we have $d_{H-S}(x) \ge \rho - 1$ for each $x \in V(T)$. Note that $|E(G[T]) \cap C| \le |T| - 1$, therefore, $\theta_H(S, T, \rho) \ge \rho + (\rho + 1 - 1)(\rho - 1 - \rho) + (\rho - \rho) - 1 = -1$, which contradicts (3). Thus we may assume $|S| \ge 2$. Since $G[T]$ is complete, we have $|E(G[T])| = |T|(|T| - 1)/2$. Note that C is a Hamiltonian cycle of G, $|E(G[T]) \cap C| \le |T| - 1$. We obtain

$$\begin{aligned} \sum_{x\in T} d_{H-S}(x) &\ge 2|E(G[T]) \setminus E(C)| \\ &\ge |T|(|T| - 1) - 2(|T| - 1) = (|T| - 1)(|T| - 2). \end{aligned}$$

Noting that $|S| \ge 2$ and $|T| = \rho + 1$, we have

$$\theta_H(S, T, \rho) \ge \rho|S| + (|T| - 1)(|T| - 2) - \rho|T| - h_H(S, T, \rho) \ge 2\rho - 2\rho - 1 = -1$$

which contradicts (3).

Now, we complete the proof of Theorem 6. □

3. Concluding remarks

Remark 1. Note that Theorem 4 cannot show the existence of a regular factor in $G - E(C)$. Consider a graph $G = K_t \oplus K_{2\alpha} \oplus K_{t+2\alpha}$ which is used in [6] where t is a sufficiently large integer. Then G satisfies the Ore-type condition and $G - E(C)$ is connected for any Hamiltonian cycle C. Note that $\delta_{G-E(C)} \ge k - 2$. For each $x \in V(G)$, we have

$$d(x) = \begin{cases} t + 2\alpha - 1, & \text{if } x \in V(K_t), \\ n - 1, & \text{if } x \in V(K_{2\alpha}), \\ t + 4\alpha - 1, & \text{if } x \in V(K_{t+2\alpha}). \end{cases}$$

However, $G - E(C)$ has two vertices $x, y \in V(K_t)$ such that

$$d_{G-E(C)}(x) + d_{G-E(C)}(y) = 2(|K_t| + |K_{2\alpha}|) - 2 - 4 = 2(t + 2\alpha) - 6 = n - 6.$$

Hence we cannot apply Theorem 3 to $G - E(C)$ to ensure the existence of $(k-2)$-factor. In particular, for any two nonadjacent vertices x and y of $V(K_t)$, $\max\{d_{G-E(C)}(x), d_{G-E(C)}(y)\} < \frac{n}{2}$. Consequently, we cannot apply Theorem 4 to $G - E(C)$ to ensure the existence of $(k - 2)$-factor. Note that G satisfies the condition of Theorem 6. So our theorem (Theorem 6) guarantees that G has a k-factor but Theorems 3 and 4 do not.

Remark 2. In the proof of Theorem 6, for each Hamiltonian cycle C in G, the requirement of connectedness of $G - E(C)$ is necessary. For example, for odd k, consider the graph G constructed by joining two complete graphs K_{2m-3} and K_{2m+3} with two independent edges e_1 and e_2, where m is a sufficiently large integer. Then G is 2-connected, $G - E(C)$ is disconnected, $|G| = 4m$ and

$$\max\{d_G(x), d_G(y)\} \geq 2m + 2 = \frac{|G| + 4}{2}$$

for any two nonadjacent vertices x and y of $V(G)$. For any Hamiltonian cycle C, $G-E(C)$ has no $(k-2)$-factor since C contains both e_1 and e_2, and thus $G - E(C)$ consists of two components of odd order. Clearly, G satisfies all the conditions of the main theorem (Theorem 6) but has no k-factor containing a Hamiltonian cycle (by Theorem 7).

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